JOURNAL OF MATERIALS SCIENCE **39** (2004) 2811–2820

Electromechanical interaction of linear piezoelectric materials with a surface electrode

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This paper presents a study on the electromechanical interaction between a compliant surface electrode and a semi-infinite piezoelectric material. Using integral transforms, general expressions of stresses, displacements, electric potential and electric displacement have been obtained for symmetrically deformed electrode. For surface electrode subjected to uniform displacement, the normal stress at the edges of the electrode displays a square root singularity. Such a stress singularity will eventually induce the initiation of crack and introduce mechanical and electric instability.

1. Introduction

Smart structures using smart materials have potential applications in many areas, especially in controlling motion that is related to structural deformation. In these smart materials, piezoelectric materials, shape memory alloys, electrostrictive materials, and magnetostrive materials have been widely used in electromechanical actuators and sensors. The electromechanical coupling in piezoelectric materials provides a fundamental mechanism for sensing mechanical disturbances from the measurements of induced electric potentials, and for controlling structural behavior via external electric loading. Among piezoelectric materials, piezoelectric ceramics have been used as displacement actuators because of their high piezoelectric performance. Such actuators rely on the concept of multiplayer, in which electrodes alternate with ceramic layers. The electrodes are in contact with piezoelectric ceramics and are used as driving components to control the motion of the multiplayer stacks. A singularity of electric field and stress field could be created due to the electroelastic interaction between the electrode and the layer of piezoelectric material, which could lead to the nucleation and propagation of cracks for relaxing the incompatible strains.

To understand the fracture processes in piezoelectric ceramics and improve the device design, linear electroelastic fracture mechanics (LEEFM) has been established to study the cracking behavior of piezoelectric materials [1–11]. Many important achievements have been made in the last several decades, while the study on the electroelastic interaction due to the contact between electrode and piezoelectric material is at early stage. The knowledge of the electromechanical interaction is essential for understanding the reliability of piezoelectric components used in microelectromechanical systems and microtransducers. Recently, Castro and Sosa [12] studied a two-dimension electromechanical coupling problem on a semi-infinite dielectric solid. Shindo *et al*. [13] analyzed electric and stress fields inside a semi-infinite piezoelectric medium. Ye and He [14] considered the singularity of fields in a piezoelectric layer. However, none of the works considers the dielectric effect due to the surrounding medium, such as air, which alters the distribution of electric charge over the surface of piezoelectric medium.

The purpose of this work is to revisit the electromechanical interaction due to a surface electrode attached to the surface of a semi-infinite piezoelectric material. The piezoelectric material is subjected to general mechanical and electric loading over the electrode. The classic electric boundary conditions on the surface of piezoelectric materials are used, which takes account of the effect of the surrounding dielectric medium on the deformation of the material. The Fourier transforms are used to reduce the problem to the solution of a set of dual integral equations, from which closed-form solutions are obtained.

2. Fundamental electromechanical equations

Consider a linear piezoelectric material, the governing equations in the Cartesian coordinates x_i ($i = 1, 2, 3$) are given by

$$\sigma_{\mathrm{ij,i}} = 0 \quad \text{and} \quad D_{\mathrm{i,i}} = 0 \tag{1}$$

where σ_{ij} is stress tensor, D_i is electric displacement vector, a comma denotes partial differentiation with respect to the coordinate x_i, and the Einstein summation convention over repeated indices is used. For an anisotropic piezoelectric material, the constitutive relations are

$$\sigma_{\mathrm{ij}} = c_{\mathrm{ijkl}}\varepsilon_{\mathrm{kl}} - e_{\mathrm{kij}}E_\mathrm{k} \quad \text{and} \quad D_\mathrm{i} = e_{\mathrm{ikl}}\varepsilon_{\mathrm{kl}} + \in_{\mathrm{ik}} E_\mathrm{k} \tag{2}$$

where $\varepsilon_{\mathrm{ij}}$ is strain tensor, E_i is electric field intensity, c_{ijkl} is elastic stiffness tensor measured in a constant electric field intensity, e_{ikl} is piezoelectric tensor measured in a spontaneous electric field, and $\in_{ik}$ is dielectric tensor. Crystal symmetry places restrictions among the elements of any tensor that characterizes the material properties of a crystal. The interchange symmetry of the tensors gives

$$\begin{aligned} &c_{ijkl} = c_{ijlk} = c_{jikl} = c_{jilk} = c_{klij}, \\ &e_{kij} = e_{kji}, \quad \text{and} \quad \in_{ij} = \in_{ji} \end{aligned} \tag{3}$$

The relation between the strain tensor and displacement, u_i, is

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{4}$$

and the electric field intensity is related to electric potential, ϕ, as

$$E_i = -\phi_{,i} \tag{5}$$

In air, the electric potential satisfies the following equation

$$\phi^a_{,ii} = 0 \tag{6}$$

with

$$D^a_i = \in_0 E^a_i \tag{7}$$

which represents the relation between the electric displacement and the electric field intensity, where $\in_0$ is the permittivity of free space (Note that the dielectric constant of air ($\in_r$) is 1). Here the superscript a denotes the field variables in air.

We consider a transversely isotropic piezoelectric material of the hexagonal crystal class 6 mm. The constitutive relations are

$$\begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{11} & c_{13} & 0 & 0 & 0 \\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{pmatrix} \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{pmatrix} - \begin{pmatrix} 0 & 0 & e_{31} \\ 0 & 0 & e_{31} \\ 0 & 0 & e_{33} \\ 0 & e_{15} & 0 \\ e_{15} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix} \tag{8}$$

$$\begin{pmatrix} D_1 \\ D_2 \\ D_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & e_{15} & 0 \\ 0 & 0 & 0 & e_{15} & 0 & 0 \\ e_{31} & e_{31} & e_{33} & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{pmatrix} + \begin{pmatrix} \in_{11} & 0 & 0 \\ 0 & \in_{11} & 0 \\ 0 & 0 & \in_{33} \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix}$$

where

$$\begin{aligned} &\begin{pmatrix} \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 & \sigma_5 & \sigma_6 \\ \varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \varepsilon_4 & \varepsilon_5 & \varepsilon_6 \end{pmatrix} \\ &= \begin{pmatrix} \sigma_{11} & \sigma_{22} & \sigma_{33} & \sigma_{23} & \sigma_{13} & \sigma_{12} \\ \varepsilon_{11} & \varepsilon_{22} & \varepsilon_{33} & 2\varepsilon_{23} & 2\varepsilon_{13} & 2\varepsilon_{12} \end{pmatrix} \\ &c_{11} = c_{1111} = c_{2222}, c_{12} = c_{1122}, c_{13} = c_{1133} = c_{2233}, \\ &c_{33} = c_{3333}, c_{44} = c_{2323} = c_{3131} \\ &c_{66} = c_{1212} = \frac{1}{2}(c_{11} - c_{22}), e_{31} = e_{311} = e_{322}, \\ &e_{33} = e_{333}, e_{15} = e_{113} = e_{223} \end{aligned} \tag{9}$$

3. General solution of two-dimensional problems in piezoelectric materials

For two dimensional piezoelectric coupling problems in plane strain, the governing equations become

$$\begin{aligned} &c_{11}\frac{\partial^2 u_1}{\partial x_1^2} + c_{44}\frac{\partial^2 u_1}{\partial x_3^2} + (c_{13} + c_{44})\frac{\partial^2 u_3}{\partial x_1 \partial x_3} \\ &\quad + (e_{31} + e_{15})\frac{\partial^2 \phi}{\partial x_1 \partial x_3} = 0 \\ &(c_{13} + c_{44})\frac{\partial^2 u_1}{\partial x_1 \partial x_3} + c_{44}\frac{\partial^2 u_3}{\partial x_1^2} + c_{33}\frac{\partial^2 u_3}{\partial x_3^2} \\ &\quad + e_{15}\frac{\partial^2 \phi}{\partial x_1^2} + e_{33}\frac{\partial^2 \phi}{\partial x_3^2} = 0 \\ &(e_{31} + e_{15})\frac{\partial^2 u_1}{\partial x_1 \partial x_3} + e_{15}\frac{\partial^2 u_3}{\partial x_1^2} + e_{33}\frac{\partial^2 u_3}{\partial x_3^2} \\ &\quad - \in_{11} \frac{\partial^2 \phi}{\partial x_1^2} - \in_{33} \frac{\partial^2 \phi}{\partial x_3^2} = 0 \end{aligned} \tag{10}$$

which can be expressed as

$$[D]\begin{pmatrix} u_1 \\ u_3 \\ \phi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{11}$$

Here the operator $[D]$ is defined as

$$\begin{aligned} &[D] \\ &= \begin{pmatrix} c_{11}\frac{\partial^2}{\partial x_1^2} + c_{44}\frac{\partial^2}{\partial x_3^2} & (c_{13} + c_{44})\frac{\partial^2}{\partial x_1 \partial x_3} & (e_{31} + e_{15})\frac{\partial^2}{\partial x_1 \partial x_3} \\ (c_{13} + c_{44})\frac{\partial^2}{\partial x_1 \partial x_3} & c_{44}\frac{\partial^2}{\partial x_1^2} + c_{33}\frac{\partial^2}{\partial x_3^2} & e_{15}\frac{\partial^2}{\partial x_1^2} + e_{33}\frac{\partial^2}{\partial x_3^2} \\ (e_{31} + e_{15})\frac{\partial^2}{\partial x_1 \partial x_3} & e_{15}\frac{\partial^2}{\partial x_1^2} + e_{33}\frac{\partial^2}{\partial x_3^2} & -\left(\in_{11}\frac{\partial^2}{\partial x_1^2} + \in_{33}\frac{\partial^2}{\partial x_3^2}\right) \end{pmatrix} \end{aligned} \tag{12}$$

The determinant of $[D]$ is

$$\det[D] = a\frac{\partial^6}{\partial x_3^6} + b\frac{\partial^6}{\partial x_3^4 \partial x_1^2} + c\frac{\partial^6}{\partial x_3^2 \partial x_1^4} + d\frac{\partial^6}{\partial x_1^6} \quad (13)$$

in which

$$\begin{aligned}
a &= -c_{44}\left(e_{33}^2 + c_{33} \in_{33}\right)\\
b &= \left[2e_{33}c_{13}(e_{31}+e_{15}) + \in_{33}(c_{13}+c_{44})^2\right.\\
&\quad - c_{11}\left(e_{33}^2 + c_{33} \in_{33}\right) - c_{44}(c_{33} \in_{11} + c_{44} \in_{33}\\
&\quad \left. - 2e_{33}e_{31}) - c_{33}(e_{31}+e_{15})^2\right]\\
c &= \left[2e_{15}(e_{31}+e_{15})(c_{13}+c_{44}) + \in_{11}(c_{13}+c_{44})^2\right.\\
&\quad - c_{11}(c_{44} \in_{33} + c_{33} \in_{11} + 2e_{33}e_{15})\\
&\quad \left. - c_{44}\left[e_{15}^2 + c_{44} \in_{11} + (e_{31}+e_{15})^2\right]\right]\\
d &= -c_{11}\left(e_{15}^2 + c_{44} \in_{11}\right) \qquad (14)
\end{aligned}$$

Based on the cofactors Δ_{ij} of det $[D]$ $(i, j = 1, 2, 3)$, the general solutions of Equation 11 are

$$\begin{pmatrix} u_1 \\ u_3 \\ \phi \end{pmatrix} = \begin{pmatrix} \Delta_{11} & \Delta_{21} & \Delta_{31} \\ \Delta_{12} & \Delta_{22} & \Delta_{32} \\ \Delta_{13} & \Delta_{23} & \Delta_{33} \end{pmatrix} \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} \quad (15)$$

with F_i $(i = 1, 2, 3)$ satisfying the equation

$$\det[D] \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad (16)$$

Obviously, $(F_1, 0, 0)$, $(0, F_2, 0)$ and $(0, 0, F_3)$ are the solutions of Equation 15. For the problems symmetric about x_3-axis, we only consider the general solution $(0, F, 0)$ in the following analysis, in which F is a function to be determined. The corresponding cofactors, $(\Delta_{21}, \Delta_{22}, \Delta_{23})$, are

$$\begin{aligned}
\Delta_{21} &= \alpha_1 \frac{\partial^4}{\partial x_1^3 \partial x_3} + \alpha_2 \frac{\partial^4}{\partial x_1 \partial x_3^3}\\
\Delta_{22} &= -c_{11} \in_{11} \frac{\partial^4}{\partial x_1^4} - \alpha_3 \frac{\partial^4}{\partial x_1^2 \partial x_3^2} - c_{44} \in_{33} \frac{\partial^4}{\partial x_3^4}\\
\Delta_{23} &= -c_{11}e_{15}\frac{\partial^4}{\partial x_1^4} - \alpha_4 \frac{\partial^4}{\partial x_1^2 \partial x_3^2} - c_{44}e_{33}\frac{\partial^4}{\partial x_3^4} \qquad (17)
\end{aligned}$$

where

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} = \begin{pmatrix} (c_{13}+c_{44}) \in_{11} + (e_{15}+e_{31})e_{15} \\ (c_{13}+c_{44}) \in_{33} + (e_{15}+e_{31})e_{33} \\ c_{11} \in_{33} + c_{44} \in_{11} + (e_{15}+e_{31})^2 \\ c_{13}e_{33} - c_{13}(e_{15}+e_{31}) - c_{44}e_{31} \end{pmatrix} \quad (18)$$

Using the symmetry on x_3-axis, one can take the cosine Fourier transform and express F as

$$F = \frac{2}{\pi}\int_0^\infty f(\xi, x_3)\cos(x_1\xi)\,d\xi \quad (19)$$

Equation 16 gives

$$a\frac{d^6 f}{dx_3^6} - b\xi^2\frac{d^4 f}{dx_3^4} + c\xi^4\frac{d^2 f}{dx_3^2} - d\xi^6 = 0 \quad (20)$$

which is a homogeneous equation. The solution of f is a function of $\exp(\lambda\xi x_3)$, in which λ are the roots of the characteristic equation

$$a\lambda^6 - b\lambda^4 + c\lambda^2 - d = 0 \quad (21)$$

Here we only consider the lower plane $(x_3 > 0)$, in which the fields approach constant values as $x_3 \to \infty$. Depending on the properties of λ^2, there are four different general solutions of function f.

a) $\lambda_1^2 \neq \lambda_2^2 \neq \lambda_3^2 > 0$

$$f = \beta_1 e^{-\lambda_1\xi x_3} + \beta_2 e^{-\lambda_2\xi x_3} + \beta_3 e^{-\lambda_3\xi x_3} \quad (22)$$

b) $\lambda_1^2 \neq \lambda_2^2 = \lambda_3^2 > 0$

$$f = \beta_1 e^{-\lambda_1\xi x_3} + \beta_2 e^{-\lambda_2\xi x_3} + \beta_3 \xi x_3 e^{-\lambda_2\xi x_3} \quad (23)$$

c) $\lambda_1^2 = \lambda_2^2 = \lambda_3^2 > 0$

$$f = \beta_1 e^{-\lambda_1\xi x_3} + \beta_2 \xi x_3 e^{-\lambda_1\xi x_3} + \beta_3 \xi^2 x_3^2 e^{-\lambda_1\xi x_3} \quad (24)$$

d) $\lambda_1^2 > 0$ and $\lambda_2^2, \lambda_3^2 < 0$ or λ_2^2 and λ_3^2 being a pair of conjugate complex roots

In this case, the λ_2 and λ_3 are a pair of conjugate complexes $-\delta \pm i\omega$. The solution of f is

$$\begin{aligned}
f &= \beta_1 e^{-\lambda_1\xi x_3} + \beta_2 e^{-\delta\xi x_3}\cos\omega\xi x_3\\
&\quad + \beta_3 e^{-\delta\xi x_3}\sin\omega\xi x_3 \qquad (25)
\end{aligned}$$

where δ and $\omega > 0$ and β_i $(i = 1, 2, 3)$ is a function of ξ to be determined by the boundary conditions.

Using Equations 22–25 and 15 and 17, the displacement, stresses, electric and potential fields for the problems symmetric about x_3-axis can be readily obtained. The expressions of the displacement, stresses, electric and potential fields are given in Appendix A.

4. Two-dimensional contact problem

Consider a compliant electrode of length $2a_0$ being attached to the surface of a semi-infinite piezoelectric medium $(x_3 > 0)$ as shown in Fig. 1, in which the axis ox_3 is the axis of the hexagonal symmetry of the

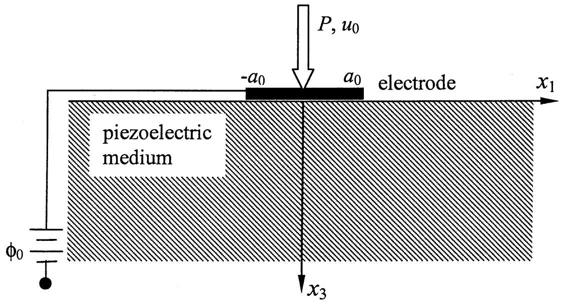

Figure 1 Schematic diagram of a piezoelectric material with a rigid electrode under mechanical and electric loading.

piezoelectric crystal. External normal load and electrical potential are applied to the electrode. The electric boundary conditions at the interface between the piezoelectric medium and air are

$$\phi(x_1, 0^+) = \phi^a(x_1, 0^-) \quad \text{and}$$
$$D_3(x_1, 0^+) = D_3^a(x_1, 0^-) \quad \text{for } |x_1| > a_0 \quad (26)$$

The stress boundary conditions are

$$\sigma_{13}(x_1, 0) = 0 \quad (27)$$

for the frictionless boundary condition, and

$$\sigma_{33}(x_1, 0) = 0 \quad \text{for } |x_1| > a_0 \quad (28)$$

In the contact zone between the electrode and the surrounding mediums, the electric boundary conditions are

$$\phi(x_1, 0^+) = \phi^a(x_1, 0^-) = \phi_0 \quad \text{for } |x_1| < a_0 \quad (29)$$

and the displacement boundary is

$$u_3(x_1, 0) = u_0(x_1) \quad \text{for } |x_1| < a_0 \quad (30)$$

where $u_0(x_1)$ is the surface displacement in the contact zone between the electrode and the piezoelectric medium.

Applying the symmetry on x_3-axis and the condition $\phi^a \to 0$, as $x_3 \to -\infty$, one can express the electric potential distribution in the air as

$$\phi^a = \frac{2}{\pi}\int_0^\infty \beta_a e^{\xi x_3} \cos x_1\xi \mathrm{d}\xi \quad \text{for } x_3 < 0 \quad (31)$$

where β_a is a constant to be determined.

Using the boundary conditions (26–30) and the field distributions given in Appendix A and Equation 31, one obtains the following equations,

(i) for the shear stress

$$b_{11}\beta_1 + b_{12}\beta_2 + b_{13}\beta_3 = 0 \quad (32)$$

(ii) for the electric potential

$$(b_{21}\beta_1 + b_{22}\beta_2 + b_{23}\beta_3)\xi^4 = \beta_a \quad (33)$$

$$\frac{2}{\pi}\sum_{i=1}^{3} b_{2i}\int_0^\infty \beta_i\xi^4\cos(\xi x_1)\,\mathrm{d}\xi = \phi_0 \quad \text{for } |x_1| < a_0 \quad (34)$$

(iii) for the normal stress

$$\frac{2}{\pi}\sum_{i=1}^{3} b_{3i}\int_0^\infty \beta_i\xi^5\cos(\xi x_1)\,\mathrm{d}\xi = 0 \quad \text{for } |x_1| > a_0 \quad (35)$$

(iv) for the displacement

$$\frac{2}{\pi}\sum_{i=1}^{3} b_{4i}\int_0^\infty \beta_i\xi^4\cos(\xi x_1)\,\mathrm{d}\xi = -u_0(x_1) \quad \text{for } |x_1| < a_0 \quad (36)$$

(v) for the electric displacement

$$\sum_{i=1}^{3} b_{5i}\int_0^\infty \beta_i\xi^5\cos(\xi x_1)\,\mathrm{d}\xi = -\epsilon_0\int_0^\infty \beta_a\xi\cos(\xi x_1)\,\mathrm{d}\xi \quad \text{for } |x_1| > a_0 \quad (37)$$

where b_{ij} ($i = 1, 2, 3, 4, 5$ and $j = 1, 2, 3$) as given in Appendix B are constants depending on the material properties of piezoelectric materials.

Using Equations 32 and 33, one obtains

$$\beta_a = -\frac{b_{23}b_{11} - b_{13}b_{21}}{b_{13}}\beta_1\xi^4 - \frac{b_{23}b_{12} - b_{13}b_{22}}{b_{13}}\beta_2\xi^4 \quad (38)$$

$$\beta_3 = -\frac{b_{11}}{b_{13}}\beta_1 - \frac{b_{12}}{b_{13}}\beta_2 \quad (39)$$

Substituting Equations 38 and 39 into Equations 34–37, we have

$$\int_0^\infty (A_{11}\beta_1 + A_{12}\beta_2)\xi^5\cos(\xi x_1)\,\mathrm{d}\xi = 0 \quad \text{for } |x_1| > a_0 \quad (40)$$

for the normal stress,

$$\int_0^\infty (A_{21}\beta_1 + A_{22}\beta_2)\xi^4\cos(\xi x_1)\,\mathrm{d}\xi = -u_0(x_1) \quad \text{for } |x_1| < a_0 \quad (41)$$

for the displacement,

$$\int_0^\infty (C_{11}\beta_1 + C_{12}\beta_2)\xi^5\cos(\xi x_1)\,\mathrm{d}\xi = 0 \quad \text{for } |x_1| > a_0 \quad (42)$$

for the electric displacement, and

$$\int_0^\infty (C_{21}\beta_1 + C_{22}\beta_2)\xi^4 \cos(\xi x_1)\,\mathrm{d}\xi = \phi_0 \quad \text{for } |x_1| > a_0 \tag{43}$$

for the electric potential. Here

$$A_{11} = b_{31}b_{13} - b_{11}b_{33} \quad \text{and} \quad A_{12} = b_{32}b_{13} - b_{12}b_{33}$$
$$A_{21} = \frac{2}{\pi}\cdot\frac{b_{41}b_{13} - b_{11}b_{43}}{b_{13}} \quad \text{and}$$
$$A_{22} = \frac{2}{\pi}\cdot\frac{b_{42}b_{13} - b_{12}b_{43}}{b_{13}}$$
$$C_{11} = b_{51}b_{13} - b_{53}b_{11} + \epsilon_0\,(b_{21}b_{13} - b_{23}b_{11}) \tag{44}$$
$$C_{12} = b_{52}b_{13} - b_{53}b_{12} + \epsilon_0\,(b_{22}b_{13} - b_{23}b_{12})$$
$$C_{21} = \frac{2}{\pi}\cdot\frac{b_{21}b_{13} - b_{11}b_{23}}{b_{13}} \quad \text{and}$$
$$C_{22} = \frac{2}{\pi}\cdot\frac{b_{22}b_{13} - b_{12}b_{23}}{b_{13}}$$

Due to the electromechanical coupling in the piezoelectric material, the displacement and electric displacement are dependent on the electrical potential and mechanical loading. This suggests,

$$A_{11}C_{12} - A_{12}C_{11} \neq 0 \quad \text{and} \quad A_{21}C_{22} - A_{22}C_{21} \neq 0 \tag{45}$$

(For decoupling problem such as the problems of elastic contact, Equations 40–43 reduces to a set of decoupled dual integral equations immediately.) Thus, the coupling dual integral Equations 40–43 can be reduced to the following decoupled dual integral equations.

$$\int_0^\infty \beta_1\xi^4 \cos(\xi x_1)\,\mathrm{d}\xi = F_1(x_1) \quad \text{for } |x_1| < a_0 \tag{46}$$
$$\int_0^\infty \beta_1\xi^5 \cos(\xi x_1)\,\mathrm{d}\xi = 0 \quad \text{for } |x_1| > a_0 \tag{47}$$

and

$$\int_0^\infty \beta_2\xi^4 \cos(\xi x_1)\,\mathrm{d}\xi = F_2(x_1) \quad \text{for } |x_1| < a_0 \tag{48}$$
$$\int_0^\infty \beta_2\xi^5 \cos(\xi x_1)\,\mathrm{d}\xi = 0 \quad \text{for } |x_1| > a_0 \tag{49}$$

where

$$F_1(x_1) = -\frac{C_{12}u_0(x_1) + A_{12}\phi_0}{A_{11}C_{12} - A_{12}C_{11}} \tag{50}$$
$$F_2(x_1) = -\frac{C_{11}u_0(x_1) + A_{11}\phi_0}{A_{11}C_{12} - A_{12}C_{11}} \tag{51}$$

The dual integral Equations 46 and 47 and 48 and 49 are a special case of a pair of dual integral equations as discussed by Sneddon [15]. The complete solutions are

$$\beta_1(\xi) = \frac{2a_0}{\pi\xi^4}\left(-J_1(a_0\xi)\int_0^1 \frac{F_1(a_0y)\,\mathrm{d}y}{\sqrt{1-y^2}} + \int_0^1 \frac{\mathrm{d}t}{\sqrt{1-t^2}}\int_0^1 a_0yF_1(a_0yt)J_0(a_0y\xi)\,\mathrm{d}y\right) \tag{52}$$

$$\beta_2(\xi) = \frac{2a_0}{\pi\xi^4}\left(-J_1(a_0\xi)\int_0^1 \frac{F_2(a_0y)\,\mathrm{d}y}{\sqrt{1-y^2}} + \int_0^1 \frac{\mathrm{d}t}{\sqrt{1-t^2}}\int_0^1 a_0yF_2(a_0yt)J_0(a_0y\xi)\,\mathrm{d}y\right) \tag{53}$$

Now, consider the plane electrode subjected to a uniform displacement, $u_0(x_1) = u_0$ over the contact zone. Since both functions $F_1(x_1)$ and $F_2(x_1)$ are constants, no solution can be obtained from Equations 52 and 53. Following the two-dimensional contact problem in linear elastic theory [16], the boundary conditions Equations 29 and 30 are rewritten as

$$\frac{\partial\phi(x_1, 0^\pm)}{\partial x_1} = 0 \quad \text{and} \quad \frac{\partial u_3(x_1, 0)}{\partial x_1} = 0 \quad \text{for } |x_1| < a_0 \tag{54}$$

Using the same procedure, the Equations 46 and 48 become

$$\int_0^\infty \beta_1\xi^5 \sin(\xi x_1)\,\mathrm{d}\xi = 0 \quad \text{for } |x_1| < a_0 \tag{55}$$
$$\int_0^\infty \beta_2\xi^5 \sin(\xi x_1)\,\mathrm{d}\xi = 0 \quad \text{for } |x_1| < a_0 \tag{56}$$

Together with Equations 47 and 49, the solutions of the dual integral equations are

$$\beta_1 = \alpha_1\frac{2}{\pi}\cdot\frac{J_0(a_0\xi)}{\xi^5} \quad \text{and} \quad \beta_2 = \alpha_2\frac{2}{\pi}\cdot\frac{J_0(a_0\xi)}{\xi^5} \tag{57}$$

where α_1 and α_2 are two constants to be determined. Substituting Equation 57 into Equations 38 and 39, we obtain

$$\beta_3 = -\frac{2}{\pi}\cdot\frac{J_0(a_0\xi)}{b_{13}\xi^5}(\alpha_1 b_{11} + \alpha_2 b_{12}) \tag{58}$$
$$\beta_a = -\frac{2}{\pi}\cdot\frac{J_0(a_0\xi)}{b_{13}\xi}[\alpha_1(b_{23}b_{11} - b_{13}b_{21}) + \alpha_2(b_{23}b_{12} - b_{13}b_{22})] \tag{59}$$

In the contact zone ($|x_1| < a_0$), the normal stress is

$$\sigma_{33} = \frac{4}{\pi^2 b_{13}}(A_{11}\alpha_1 + A_{12}\alpha_2)\int_0^\infty J_0(a_0\xi)\cos(\xi x_1)\,\mathrm{d}\xi = \frac{4}{\pi^2}\cdot\frac{A_{11}\alpha_1 + A_{12}\alpha_2}{b_{13}\sqrt{a_0^2 - x_1^2}} \tag{60}$$

and the charge density is

$$\rho = \frac{4}{\pi^2 b_{13}}(D_{11}\alpha_1 + D_{12}\alpha_2)\int_0^\infty J_0(a_0\xi)\cos(\xi x_1)\,d\xi$$
$$= \frac{4}{\pi^2}\cdot\frac{D_{11}\alpha_1 + D_{12}\alpha_2}{b_{13}\sqrt{a_0^2 - x_1^2}} \quad (61)$$

where

$$D_{11} = b_{51}b_{13} - b_{53}b_{11} \quad \text{and} \quad D_{12} = b_{52}b_{13} - b_{53}b_{12} \quad (62)$$

Both the normal stress and the electric charge density demonstrate the square root singularity at both edges of the electrode. Such a stress singularity may induce the initiation of edge crack and cause the delamination of the electrode from the piezoelectric materials. Eventually, this will lead to mechanical and electric instability.

If the total load applied to the electrode is P, there is

$$P = -\int_{-a_0}^{a_0}\sigma_{33}dx_1 = -\frac{4}{\pi}\cdot\frac{A_{11}\alpha_1 + A_{12}\alpha_2}{b_{13}} \quad (63)$$

If the total electric charge stored in the electrode is Q, we have

$$Q = \int_{-a_0}^{a_0}\rho dx_1 = \frac{4}{\pi}\cdot\frac{D_{11}\alpha_1 + D_{12}\alpha_2}{b_{13}} \quad (64)$$

Equations 63 and 64 gives

$$\alpha_1 = \frac{\pi b_{13}(D_{12}P + A_{12}Q)}{4(A_{12}D_{11} - A_{11}D_{12})} \quad \text{and}$$
$$\alpha_2 = -\frac{\pi b_{13}(D_{11}P + A_{11}Q)}{4(A_{12}D_{11} - A_{11}D_{12})} \quad (65)$$

Using Equation 65, the constants $\beta_i\,(i = 1, 2, 3)$ and β_a are

$$\beta_1 = \frac{b_{13}}{2}\cdot\frac{J_0(a_0\xi)}{\xi^5}\cdot\frac{(D_{12}P + A_{12}Q)}{(A_{12}D_{11} - A_{11}D_{12})}$$
$$\beta_2 = -\frac{b_{13}}{2}\cdot\frac{J_0(a_0\xi)}{\xi^5}\cdot\frac{(D_{11}P + A_{11}Q)}{(A_{12}D_{11} - A_{11}D_{12})}$$
$$\beta_3 = -\frac{1}{2}\cdot\frac{J_0(a_0\xi)}{\xi^5} \cdot\frac{(D_{12}b_{11} - D_{11}b_{12})P + (A_{12}b_{11} - A_{11}b_{12})Q}{(A_{12}D_{11} - A_{11}D_{12})} \quad (66)$$
$$\beta_a = -\frac{1}{2}\cdot\frac{J_0(a_0\xi)}{\xi}\cdot\frac{1}{(A_{12}D_{11} - A_{11}D_{12})}[[D_{12}(b_{23}b_{11} - b_{13}b_{21}) - D_{11}(b_{23}b_{12} - b_{13}b_{22})]P + [A_{12}(b_{23}b_{11} - b_{13}b_{21}) - A_{11}(b_{23}b_{12} - b_{13}b_{22})]Q]$$

TABLE I Material properties of PZT-4 piezoelectric ceramics

Elastic constants (10^{10} N/m^2)				
c_{11}	c_{12}	c_{13}	c_{33}	c_{44}
13.9	7.78	7.43	11.3	2.56
Piezoelectric constants (C/m^2)			Dielectric permittivities (10^{-9} C/Vm)	
e_{31}	e_{33}	e_{15}	ϵ_{11}	ϵ_{33}
−6.98	13.84	13.44	6.00	5.47

Substituting Equation 65 into Equations 60 and 61, there are

$$\sigma_{33} = -\frac{1}{\pi}\cdot\frac{P}{\sqrt{a_0^2 - x_1^2}} \quad \text{and} \quad \rho = \frac{1}{\pi}\cdot\frac{Q}{\sqrt{a_0^2 - x_1^2}} \quad (67)$$

The normal stress distribution in the contact zone is the same as that for the two-dimension plane-contact problem [16] and is independent of the electric voltage on the electrode. On the other side, the electric charge in the electrode does not depend on the load applied to the electrode. Different from the parallel capacitor, non-uniform distribution of electric charge over the electrode is found.

5. Numerical results and discussions

Numerical results for the distribution of stresses inside a semi-infinite material are presented here for PZT-4 piezoelectric ceramics and the total electric charge in the contact zone being zero ($Q = 0$). The poling direction is assumed to be parallel to the x_3-axis. Its material properties are listed in Table I [5], where the unit of the force is in Newton (N), the unit of the electric charge in Coulomb (C), the unit of the electric voltage in volt (V), and the unit of the length in meter (m). The eigenvalues of the characteristic Equation 20 are

$$\lambda_{1,2} = \pm 1.19103,\ \lambda_{3,4} = -1.08707 \pm 0.27439i,$$
$$\lambda_{5,6} = 1.08707 \pm 0.27439i$$

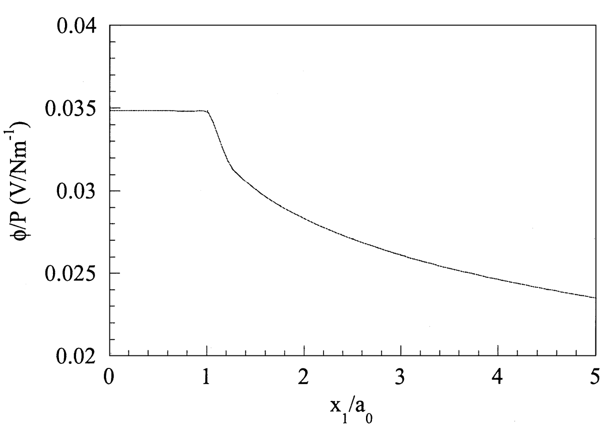

Figure 2 Electric potential distribution on the surface of the PZT-4 ceramics.

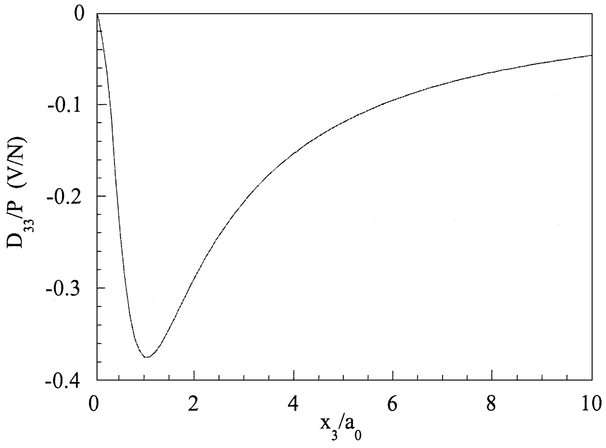

Figure 3 Electric displacement distribution underneath the electrode ($x_1 = 0$).

It is obvious that case (d) of Equation 24 does exist for piezoelectric materials. Fig. 2 shows the distribution of electric potential on the surface of the semi-infinite piezoelectric material. As expected, a uniform and nonzero electric potential in the contact zone is observed, which is induced by the external force applied to the electrode. The electric potential underneath the electrode reaches the maximum the same as that of the electrode, then it decreases with the distance away from the electrode. The distribution of the electric displacement along the x_3 axis is shown in Fig. 3. It has the value of zero at the surface of the PZT-4 ceramics, which satisfies the requirement of total electric charge at the electrode being zero ($Q = 0$). It is interesting that the electric displacement starts at zero in the contact zone, reaches the minimum at the depth of $x_3/a_0 = 1$, then gradually increases with the distance away from the electrode. This indicates that, underneath the electrode the piezoelectric material experiences the strongest electromechanical interaction at the location of $x_3/a_0 = 1$.

Fig. 4 displays the distribution of the displacement component u_1 along the x_1 axis. For the given condition, the surface of the PZT-4 ceramics near the electrode is moved toward the edges of the electrode. However, material underneath the electrode ($x_3/a_0 > 1$) is squeezed out because of the mass conservation. The normal displacement of the PZT-4 ceramics underneath the electrode is shown in Fig. 5. The PZT-4 ceramics is pushed downward. The normal displacement at $x_1 = 0$ is the same as the displacement applied to the electrode if $x_3/a_0 < 0.5$, then it starts to decreases. A dramatic drop occurs at $x_3/a_0 = 1$, which is the same as the location with the minimum electric displacement, suggesting the strongest electromechanical coupling underneath the electrode in the piezoelectric material.

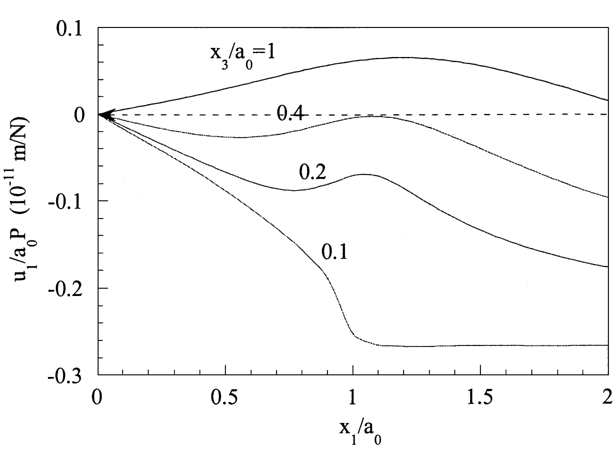

Figure 4 Distribution of displacement component u_1 along the x_1 axis.

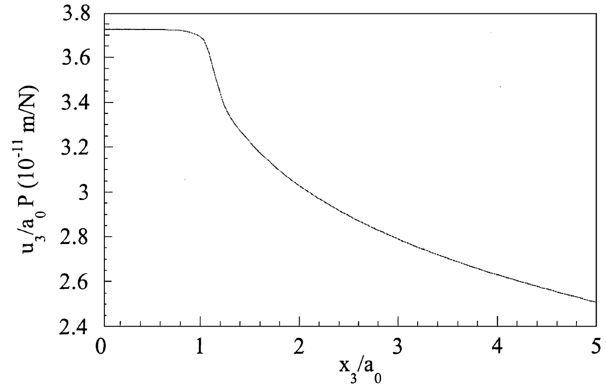

Figure 5 Distribution of displacement component u_3 along the x_3 axis at $x_1 = 0$.

Fig. 6 shows the stresses distribution underneath the electrode. The shear stress is found to be zero at $x_1 = 0$ because of the symmetry. However away from the symmetric plane, it starts at zero on the surface of the PZT-4 ceramics, decreases to the minimum at $x_1/a_0 = 0.25$, then increases. For the normal stress, the magnitude

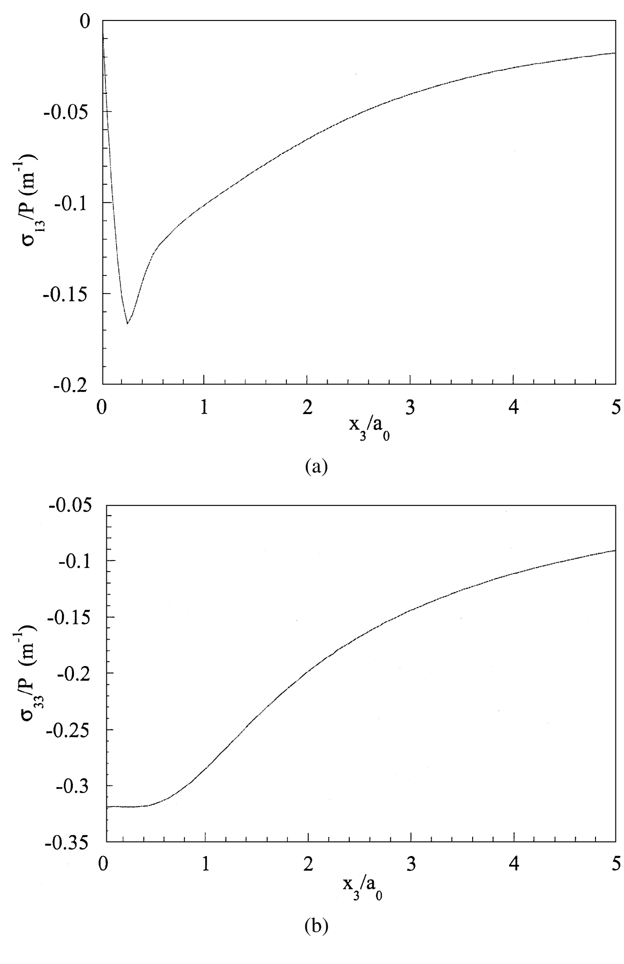

Figure 6 (a) Shear stress distribution along the x_3 direction at $x_1/a_0 = 1$, (b) Normal stress distribution along the x_3 direction at $x_1 = 0$.

starts at the maximum, then gradually decreases and approaches zero with the distance away from the electrode. The maximum stress in the piezoelectric materials occurs at the interface between the electrode and the piezoelectric material. When subjected to larger displacement, the electrode may be detached from the piezoelectric material due to the nucleation and propagation of interfacial crack. This will lead to mechanical and electric failure.

6. Conclusion

The electroelastic problem of a compliant electrode attached onto the surface of a semi-infinite piezoelectric material was studied by using the appropriate electrical boundaries and considering the effect of the surrounding dielectric medium. The dielectric medium was treated as air. General solutions for a transversely isotropic piezoelectric material of the hexagonal crystal class 6 mm have been given in the analysis. Using the general solutions, closed form solutions of field variables including electric potential, electric displacement field, displacement field, and stresses field for symmetric shape of compliant electrodes were obtained by using the Fourier transform and dual integral equations. For a plane electrode subjected to a uniform displacement, the distribution of the normal stress in the contact zone is the same as in the elastic contact theory. A square root singularity of the normal stress and charge density at the edges of the electrode was found. Such high stress at the edges of the electrode will eventually induce the initiation of crack and cause mechanical and electric failure.

Acknowledgement

The valuable comments by the reviewer are especially appreciated.

Appendix A

Based on the solution of the auxiliary function f, the displacement, stresses, electric, and potential field in the upper plane can be easily calculated by using Mathematica. Here, we give only the field functions for $\lambda_1^2 > 0$ and $\lambda_2^2, \lambda_3^2 < 0$ or λ_2^2 and λ_3^2 being a pair of conjugate complex roots ($\lambda_2, \lambda_3 = \gamma \pm i\omega$).

$$\begin{aligned} u_1 = {} & \frac{2}{\pi}\lambda_1\left(-\alpha_1+\alpha_2\lambda_1^2\right)\int_0^\infty \beta_1\xi^4 \sin(\xi x_1)e^{-\lambda_1\xi x_3}\mathrm{d}\xi \\ & + \frac{2}{\pi}\int_0^\infty \beta_2\xi^4 \sin(\xi x_1)e^{-\gamma\xi x_3}\mathrm{d}\xi \\ & \times [(-\alpha_1+\alpha_2\gamma^2-3\alpha_2\omega^2)\gamma\cos(\omega\xi x_3) \\ & -(\alpha_1-3\alpha_2\gamma^2+\alpha_2\omega^2)\omega\sin(\omega\xi x_3)] \\ & + \frac{2}{\pi}\int_0^\infty \beta_3\xi^4 \sin(\xi x_1)e^{-\gamma\xi x_3}\,\mathrm{d}\xi \\ & \times [(\alpha_1-3\alpha_2\gamma^2+\alpha_2\omega^2)\omega\cos(\omega\xi x_3) \\ & -(\alpha_1-\alpha_2\gamma^2+3\alpha_2\omega^2)\gamma\sin(\omega\xi x_3)] \qquad \text{(A1)} \end{aligned}$$

$$\begin{aligned} u_3 = {} & -\frac{2}{\pi}\left(c_{11}\in_{11}-\alpha_3\lambda_1^2+c_{44}\in_{33}\lambda_1^4\right) \\ & \times \int_0^\infty \beta_1\xi^4\cos(\xi x_1)e^{-\lambda_1\xi x_3}\mathrm{d}\xi \\ & -\frac{2}{\pi}\int_0^\infty \beta_2\xi^4\cos(\xi x_1)e^{-\gamma\xi x_3}\mathrm{d}\xi[(c_{11}\in_{11}-\alpha_3\gamma^2 \\ & +\alpha_3\omega^2+(\gamma^4-6\gamma^2\omega^2+\omega^4)c_{44}\in_{33})\cos(\omega\xi x_3) \\ & -2\gamma\omega(\alpha_3-2c_{44}\in_{33}(\gamma^2-\omega^2))\sin(\omega\xi x_3)] \\ & -\frac{2}{\pi}\int_0^\infty \beta_3\xi^4\cos(\xi x_1)e^{\lambda\xi x_3}\mathrm{d}\xi \\ & \times[2\gamma\omega(\alpha_3-2c_{44}\in_{33}(\gamma^2-\omega^2))\cos(\omega\xi x_3) \\ & +(c_{11}\in_{11}-\alpha_3\gamma^2+\alpha_3\omega^2 \\ & +(\gamma^4-6\gamma^2\omega^2+\omega^4)c_{44}\in_{33})\sin(\omega\xi x_3)] \qquad \text{(A2)} \end{aligned}$$

$$\begin{aligned} \phi = {} & -\frac{2}{\pi}\left(c_{11}e_{15}-\alpha_4\lambda_1^2+c_{44}e_{33}\lambda_1^4\right) \\ & \times\int_0^\infty \beta_1\xi^4\cos(\xi x_1)e^{-\lambda_1\xi x_3}\,\mathrm{d}\xi \\ & -\frac{2}{\pi}\int_0^\infty \beta_2\xi^4\cos(\xi x_1)e^{-\gamma\xi x_3}\,\mathrm{d}\xi \\ & \times[-2\gamma\omega(\alpha_4-2c_{44}e_{33}(\gamma^2-\omega^2))\sin(\omega\xi x_3) \\ & +(c_{11}e_{15}-\alpha_4\gamma^2+\alpha_4\omega^2+(\gamma^4-6\gamma^2\omega^2+\omega^4) \\ & \times c_{44}e_{33})\cos(\omega\xi x_3)] \\ & -\frac{2}{\pi}\int_0^\infty \beta_3\xi^4\cos(\xi x_1)e^{-\gamma\xi x_3}\,\mathrm{d}\xi \\ & \times[2\gamma\omega(\alpha_4-2c_{44}e_{33}(\gamma^2-\omega^2))\cos(\omega\xi x_3) \\ & +(c_{11}e_{15}-\alpha_4\gamma^2+\alpha_4\omega^2 \\ & +(\gamma^4-6\gamma^2\omega^2+\omega^4)\,c_{44}e_{33})\sin(\omega\xi x_3)] \qquad \text{(A3)} \end{aligned}$$

$$\begin{aligned} E_1 = {} & -\frac{2}{\pi}\left(c_{11}e_{15}-\alpha_4\lambda_1^2+c_{44}e_{33}\lambda_1^4\right) \\ & \times\int_0^\infty \beta_1\xi^5\sin(\xi x_1)e^{-\lambda_1\xi x_3}\,\mathrm{d}\xi \\ & -\frac{2}{\pi}\int_0^\infty \beta_2\xi^5\sin(\xi x_1)e^{-\gamma\xi x_3}\,\mathrm{d}\xi \\ & \times[-2\gamma\omega(\alpha_4-2c_{44}e_{33}(\gamma^2-\omega^2))\sin(\omega\xi x_3) \\ & +(c_{11}e_{15}-\alpha_4\gamma^2+\alpha_4\omega^2 \\ & +(\gamma^4-6\gamma^2\omega^2+\omega^4)\,c_{44}e_{33})\cos(\omega\xi x_3)] \\ & -\frac{2}{\pi}\int_0^\infty \beta_3\xi^5\sin(\xi x_1)e^{-\gamma\xi x_3}\,\mathrm{d}\xi \\ & \times[2\gamma\omega(\alpha_4-2c_{44}e_{33}(\gamma^2-\omega^2))\cos(\omega\xi x_3) \\ & +(c_{11}e_{15}-\alpha_4\gamma^2+\alpha_4\omega^2+(\gamma^4-6\gamma^2\omega^2+\omega^4) \\ & \times c_{44}e_{33})\sin(\omega\xi x_3)] \qquad \text{(A4)} \end{aligned}$$

$$\begin{aligned} E_3 = {} & -\frac{2}{\pi}\lambda_1\left(c_{11}e_{15}-\alpha_4\lambda_1^2+c_{44}e_{33}\lambda_1^4\right) \\ & \times\int_0^\infty \beta_1\xi^5\cos(\xi x_1)e^{-\lambda_1\xi x_3}\,\mathrm{d}\xi \\ & -\frac{2}{\pi}\int_0^\infty \beta_2\xi^5\cos(\xi x_1)e^{-\gamma\xi x_3}\,\mathrm{d}\xi \end{aligned}$$

$$\times [(c_{11}e_{15} + \alpha_4\omega^2 - 3\alpha_4\gamma^2$$
$$+ (\gamma^4 - 10\gamma^2\omega^2 + 5\omega^4)c_{44}e_{33})\omega \sin(\omega\xi x_3)$$
$$+ (c_{11}e_{15} - \alpha_4\gamma^2 + 3\alpha_4\omega^2 + (\gamma^4 - 10\gamma^2\omega^2$$
$$+ 5\omega^4)\, c_{44}e_{33})\gamma \cos(\omega\xi x_3)]$$
$$- \frac{2}{\pi}\int_0^\infty \beta_3\xi^5 \cos(\xi x_1)e^{-\gamma\xi x_3}\, \mathrm{d}\xi$$
$$\times [(c_{11}e_{15} - \alpha_4\gamma^2 + 3\alpha_4\omega^2$$
$$+ (\gamma^4 - 10\gamma^2\omega^2 + 5\omega^4)c_{44}e_{33})\gamma \sin(\omega\xi x_3)$$
$$+ (c_{11}e_{15} - \alpha_4\omega^2 + 3\alpha_4\gamma^2 - (5\gamma^4 - 10\gamma^2\omega^2$$
$$+ \omega^4)\, c_{44}e_{33})\omega \cos(\omega\xi x_3)] \qquad \text{(A5)}$$

$$\sigma_{11} = \frac{2}{\pi}\lambda_1\big[c_{11}\big(-\alpha_1 + \alpha_2\lambda_1^2\big) + c_{13}\big(c_{11} \in_{11} -\alpha_3\lambda_1^2$$
$$+ c_{44} \in_{33} \lambda_1^4\big) + e_{31}\big(c_{11}e_{15} - \alpha_4\lambda_1^2 + c_{44}e_{33}\lambda_1^4\big)\big]$$
$$\times \int_0^\infty \beta_1\xi^5 \cos(\xi x_1)e^{-\lambda_1\xi x_3}\, \mathrm{d}\xi$$
$$+ \frac{2}{\pi}\int_0^\infty \beta_2\xi^5 \cos(\xi x_1)e^{-\gamma\xi x_3}\, \mathrm{d}\xi$$
$$\times [(c_{11}\gamma(-\alpha_1 + \alpha_2\gamma^2 - 3\alpha_2\omega^2)$$
$$+ c_{13}\gamma(c_{11} \in_{11} -\alpha_3\gamma^2 + 3\alpha_3\omega^2$$
$$+ c_{44} \in_{33} (\gamma^4 - 10\gamma^2\omega^2 + 5\omega^4))$$
$$+ e_{31}\gamma(c_{11}e_{15} - \alpha_4\gamma^2 + 3\alpha_4\omega^2$$
$$+ c_{44}e_{33}(\gamma^4 - 10\gamma^2\omega^2 + 5\omega^4)))\cos(\omega\xi x_3)$$
$$+ (c_{11}\omega(-\alpha_1 + 3\alpha_2\gamma^2 - \alpha_2\omega^2)$$
$$+ c_{13}\omega(c_{11} \in_{11} -3\alpha_3\gamma^2 + \alpha_3\omega^2$$
$$+ c_{44} \in_{33} (5\gamma^4 - 10\gamma^2\omega^2 + \omega^4))$$
$$+ e_{31}\omega(c_{11}e_{15} - 3\alpha_4\gamma^2 + \alpha_4\omega^2$$
$$+ c_{44}e_{33}(5\gamma^4 - 10\gamma^2\omega^2 + \omega^4)))\sin(\omega\xi x_3)]$$
$$+ \frac{2}{\pi}\int_0^\infty \beta_3\xi^5 \cos(\xi x_1)e^{-\gamma\xi x_3}\mathrm{d}\xi$$
$$\times [(c_{11}\omega(\alpha_1 - 3\alpha_2\gamma^2 + \alpha_2\omega^2)$$
$$- c_{13}\omega(c_{11} \in_{11} -3\alpha_3\gamma^2 + \alpha_3\omega^2$$
$$+ c_{44} \in_{33} (5\gamma^4 - 10\gamma^2\omega^2 + \omega^4))$$
$$- e_{31}\omega(c_{11}e_{15} - 3\alpha_4\gamma^2 + \alpha_4\omega^2$$
$$+ c_{44}e_{33}(5\gamma^4 - 10\gamma^2\omega^2 + \omega^4)))\cos(\omega\xi x_3)$$
$$+ (c_{11}\gamma(-\alpha_1 + \alpha_2\gamma^2 - 3\alpha_2\omega^2)$$
$$+ c_{13}\gamma(c_{11} \in_{11} -\alpha_3\gamma^2 + 3\alpha_3\omega^2$$
$$+ c_{44} \in_{33} (\gamma^4 - 10\gamma^2\omega^2 + 5\omega^4))$$
$$+ e_{31}\gamma(c_{11}e_{15} - \alpha_4\gamma^2 + 3\alpha_4\omega^2$$
$$+ c_{44}e_{33}(\gamma^4 - 10\gamma^2\omega^2 + 5\omega^4)))\sin(\omega\xi x_3)] \qquad \text{(A6)}$$

$$\sigma_{33} = \frac{2}{\pi}\lambda_1[c_{13}\big(-\alpha_1 + \alpha_2\lambda_1^2\big)$$
$$+ c_{13}\big(c_{11} \in_{11} -\alpha_3\lambda_1^2 + c_{44} \in_{33} \lambda_1^4\big)$$
$$+ e_{33}\big(c_{11}e_{15} - \alpha_4\lambda_1^2 + c_{44}e_{33}\lambda_1^4\big)\big]$$
$$\times \int_0^\infty \beta_1\xi^5 \cos(\xi x_1)e^{-\lambda_1\xi x_3}\, \mathrm{d}\xi$$
$$+ \frac{2}{\pi}\int_0^\infty \beta_2\xi^5 \cos(\xi x_1)e^{-\gamma\xi x_3}\, \mathrm{d}\xi$$
$$\times [(c_{13}\gamma(-\alpha_1 + \alpha_2\gamma^2 - 3\alpha_2\omega^2)$$
$$+ c_{33}\gamma(c_{11} \in_{11} -\alpha_3\gamma^2 + 3\alpha_3\omega^2$$
$$+ c_{44} \in_{33} (\gamma^4 - 10\gamma^2\omega^2 + 5\omega^4))$$
$$+ e_{33}\gamma(c_{11}e_{15} - \alpha_4\gamma^2 + 3\alpha_4\omega^2$$
$$+ c_{44}e_{33}(\gamma^4 - 10\gamma^2\omega^2 + 5\omega^4)))\cos(\omega\xi x_3)$$
$$+ c_{33}\omega(c_{11} \in_{11} -3\alpha_3\gamma^2 + \alpha_3\omega^2$$
$$+ c_{44} \in_{33} (5\gamma^4 - 10\gamma^2\omega^2 + \omega^4))$$
$$+ e_{33}\omega(c_{11}e_{15} - 3\alpha_4\gamma^2 + \alpha_4\omega^2$$
$$+ c_{44}e_{33}(5\gamma^4 - 10\gamma^2\omega^2 + \omega^4)))\sin(\omega\xi x_3)]$$
$$+ \frac{2}{\pi}\int_0^\infty \beta_3\xi^5 \cos(\xi x_1)e^{-\gamma\xi x_3}\, \mathrm{d}\xi$$
$$\times [(c_{13}\omega(\alpha_1 - 3\alpha_2\gamma^2 + \alpha_2\omega^2)$$
$$- c_{33}\omega(c_{11} \in_{11} -3\alpha_3\gamma^2 + \alpha_3\omega^2$$
$$+ c_{44} \in_{33} (5\gamma^4 - 10\gamma^2\omega^2 + \omega^4))$$
$$- e_{33}\omega(c_{11}e_{15} - 3\alpha_4\gamma^2 + \alpha_4\omega^2$$
$$+ c_{44}e_{33}(5\gamma^4 - 10\gamma^2\omega^2 + \omega^4)))\cos(\omega\xi x_3)$$
$$+ (c_{13}\gamma(-\alpha_1 + \alpha_2\gamma^2 - 3\alpha_2\omega^2)$$
$$+ c_{33}\gamma(c_{11} \in_{11} -\alpha_3\gamma^2 + 3\alpha_3\omega^2$$
$$+ c_{44} \in_{33} (\gamma^4 - 10\gamma^2\omega^2 + 5\omega^4))$$
$$+ e_{33}\gamma(c_{11}e_{15} - \alpha_4\gamma^2 + 3\alpha_4\omega^2$$
$$+ c_{44}e_{33}(\gamma^4 - 10\gamma^2\omega^2 + 5\omega^4)))\sin(\omega\xi x_3)] \qquad \text{(A7)}$$

$$\sigma_{13} = \frac{2}{\pi}\big[-c_{44}\lambda_1^2\big(-\alpha_1 + \alpha_2\lambda_1^2\big) + c_{44}\big(c_{11} \in_{11} -\alpha_3\lambda_1^2$$
$$+ c_{44} \in_{33} \lambda_1^4\big) + e_{15}\big(c_{11}e_{15} - \alpha_4\lambda_1^2 + c_{44}e_{33}\lambda_1^4\big)\big]$$
$$\times \int_0^\infty \beta_1\xi^5 \sin(\xi x_1)e^{-\lambda_1\xi x_3}\, \mathrm{d}\xi$$
$$+ \frac{2}{\pi}\int_0^\infty \beta_2\xi^5 \sin(\xi x_1)e^{-\gamma\xi x_3}\, \mathrm{d}\xi$$
$$\times [(c_{44}\gamma^2(\alpha_1 - \alpha_2\gamma^2 + 6\alpha_2\omega^2)$$
$$+ c_{44}(c_{11} \in_{11} -\alpha_3\gamma^2 + \alpha_3\omega^2$$
$$+ c_{44} \in_{33} (\gamma^4 - 6\gamma^2\omega^2 + \omega^4)$$
$$- \omega^2(\alpha_1 + \alpha_2\omega^2)) + e_{15}(c_{11}e_{15} - \alpha_4\gamma^2 + \alpha_4\omega^2$$
$$+ c_{44}e_{33}(\gamma^4 - 6\gamma^2\omega^2 + \omega^4)))\cos(\omega\xi x_3)$$
$$+ 2\gamma\omega(c_{44}(\alpha_1 - \alpha_3 - 2\alpha_2\gamma^2 + 2\alpha_2\omega^2$$
$$+ 2c_{44} \in_{33} (\gamma^2 - \omega^2)) + e_{15}(-\alpha_4 + 2c_{44}e_{33}$$
$$\times (\gamma^2 - \omega^3)))\sin(\omega\xi x_3)]$$
$$+ \frac{2}{\pi}\int_0^\infty \beta_3\xi^5 \sin(\xi x_1)e^{-\gamma\xi x_3}\, \mathrm{d}\xi$$

$$\times [2\gamma\omega(-c_{44}(\alpha_1 - \alpha_3 - 2\alpha_2\gamma^2 + 2\alpha_2\omega^2 + 2c_{44} \in_{33} (\gamma^2 - \omega^2)) + e_{15}(\alpha_4 - 2c_{44}e_{33}(\gamma^2 - \omega^2))) \cos(\omega\xi x_3) + (c_{44}\gamma^2(\alpha_1 - \alpha_2\gamma^2 + 6\alpha_2\omega^2) + c_{44}(c_{11} \in_{11} - \alpha_3\gamma^2 + \alpha_3\omega^2 + c_{44} \in_{33} \times (\gamma^4 - 6\gamma^2\omega^2 + \omega^4) - \omega^2(\alpha_1 + \alpha_2\omega^2)) + e_{15}(c_{11}e_{15} - \alpha_4\gamma^2 + \alpha_4\omega^2 + c_{44}e_{33} \times (\gamma^4 - 6\gamma^2\omega^2 + \omega^4))) \sin(\omega\xi x_3)] \quad \text{(A8)}$$

Appendix B

Based on the solution of the auxiliary function f, constants c can be obtained from Appendix A by setting $x_3 = 0$. Here, only the constants $x_3 = 0$ for $\lambda_1^2 > 0$ and λ_2^2, $\lambda_3^2 < 0$ or λ_2^2 and λ_3^2 being a pair of conjugate complex roots ($\lambda_2, \lambda_3 = \gamma \pm i\omega$) are given as follows:

$$b_{11} = c_{44}\lambda_1^2(\alpha_1 - \alpha_2\lambda_1^2) + c_{44}(c_{11} \in_{11} - \alpha_3\lambda_1^2 + c_{44} \in_{33} \lambda_1^4) + e_{15}(c_{11}e_{15} - \alpha_4\lambda_1^2 + c_{44}e_{33}\lambda_1^4) \quad \text{(B1)}$$

$$b_{12} = c_{44}\gamma^2(\alpha_1 - \alpha_2\gamma^2 + 3\alpha_2\omega^2) + c_{44}[c_{11} \in_{11} - \alpha_3(\gamma^2 - \omega^2) + c_{44} \in_{33} (\gamma^4 + \omega^4) - 6c_{44} \in_{33} \gamma^2\omega^2] - c_{44}\omega^2(\alpha_1 - 3\alpha_2\gamma^2 + \alpha_2\omega^2) + e_{15}[c_{11}e_{15} - \alpha_4(\gamma^2 - \omega^2) + c_{44}e_{33}(\gamma^4 + \omega^4) - 6c_{44}e_{33}\gamma^2\omega^2]$$

$$b_{13} = -2c_{44}\gamma\omega[\alpha_1 + \alpha_3 - 2\alpha_2\gamma^2 + 2\alpha_2\omega^2 - 2c_{44} \in_{33} (\gamma^2 - \omega^2)] + 2e_{15}\gamma\omega[\alpha_4 - 2c_{44}e_{33}(\gamma^4 - \omega^4)]$$

$$b_{21} = -c_{11}e_{15} + \alpha_4\lambda_1^2 - c_{44}e_{33}\lambda_1^4 \quad \text{(B2)}$$

$$b_{22} = c_{11}e_{15} - \alpha_4(\gamma^2 - \omega^2) + c_{44}(\gamma^4 - 6e_{33}\gamma^2\omega^2 + e_{33}\omega^4)$$

$$b_{23} = 2\gamma\omega[\alpha_4 - 2c_{44}e_{33}(\gamma^2 - \omega^2)]$$

$$b_{31} = c_{13}\lambda_1(-\alpha_1 + \alpha_2\lambda_1^2) + c_{33}\lambda_1(c_{11} \in_{11} - \alpha_3\lambda_1^2 + c_{44} \in_{33} \lambda_1^4) + e_{33}\lambda_1(c_{11}e_{15} - \alpha_4\lambda_1^2 + c_{44}e_{33}\lambda_1^4) \quad \text{(B3)}$$

$$b_{32} = -c_{13}\gamma(\alpha_1 - \alpha_2\gamma^2 + 3\alpha_2\omega^2) + c_{33}\gamma[c_{11} \in_{11} - \alpha_3(\gamma^2 - \omega^2) + c_{44} \in_{33} (\gamma^4 + \omega^4) - 6c_{44} \in_{33} \gamma^2\omega^2] + e_{33}\gamma[c_{11}e_{15} - \alpha_4(\gamma^2 - 3\omega^2) + c_{44}e_{33}(\gamma^4 + 5\omega^4) - 10c_{44}e_{33}\gamma^2\omega^2] + 2c_{33}\gamma\omega^2[\alpha_3 - 2c_{44} \in_{33} (\gamma^2 - \omega^2)]$$

$$b_{33} = c_{13}\omega(\alpha_1 - 3\alpha_2\gamma^2 + \alpha_2\omega^2) - c_{33}\omega[c_{11} \in_{11} - \alpha_3(\gamma^2 - \omega^2) + c_{44} \in_{33} (\gamma^4 + \omega^4) - 6c_{44} \in_{33} \gamma^2\omega^2] - e_{33}\gamma[c_{11}e_{15} - \alpha_4(3\gamma^2 - \omega^2) + c_{44}e_{33}(5\gamma^4 + \omega^4) - 10c_{44}e_{33}\gamma^2\omega^2] + 2c_{33}\gamma^2\omega[\alpha_3 - 2c_{44} \in_{33} (\gamma^2 - \omega^2)]$$

$$b_{41} = c_{11} \in_{11} - \alpha_3\lambda_1^2 + c_{44} \in_{33} \lambda_1^4 \quad \text{(B4)}$$

$$b_{42} = c_{11} \in_{11} - \alpha_3(\gamma^2 - \omega^2) + c_{44} \in_{33} (\gamma^4 + \omega^4) - 6c_{44} \in_{33} \gamma^2\omega^2$$

$$b_{43} = 2\gamma\omega[\alpha_3 - 2c_{44} \in_{33} \omega^2(\gamma^2 - \omega^2)]$$

$$b_{51} = \lambda_1(-\alpha_1 + \alpha_2\lambda_1^2) + \lambda_1(c_{11} \in_{11} - \alpha_3\lambda_1^2 + c_{44} \in_{33} \lambda_1^4) - \in_{33} \lambda_1(c_{11}e_{15} - \alpha_4\lambda_1^2 + c_{44}e_{33}\lambda_1^4) \quad \text{(B5)}$$

$$b_{52} = -e_{31}\gamma(\alpha_1 - \alpha_2\gamma^2 + 3\alpha_2\omega^2) + e_{33}\gamma[c_{11} \in_{11} - \alpha_3(\gamma^2 - \omega^2) + c_{44} \in_{33} (\gamma^4 + \omega^4) - 6c_{44} \in_{33} \gamma^2\omega^2] - \in_{33} \gamma[c_{11}e_{15} - \alpha_4(\gamma^2 - 3\omega^2) + c_{44}e_{33}(\gamma^4 + 5\omega^4) - 10c_{44}e_{33}\gamma^2\omega^2] + 2e_{33}\gamma\omega^2[\alpha_3 - 2c_{44} \in_{33} (\gamma^2 - \omega^2)]$$

$$b_{53} = e_{31}\omega(\alpha_1 - 3\alpha_2\gamma^2 + \alpha_2\omega^2) + e_{33}\omega[c_{11} \in_{11} - \alpha_3(\gamma^2 - \omega^2) + c_{44} \in_{33} (\gamma^4 + \omega^4) - 6c_{44} \in_{33} \gamma^2\omega^2] + \in_{33} \omega[c_{11}e_{15} - \alpha_4(3\gamma^2 - \omega^2) + c_{44}e_{33}(5\gamma^4 + \omega^4) - 10c_{44}e_{33}\gamma^2\omega^2] + 2e_{33}\gamma^2\omega[\alpha_3 - 2c_{44} \in_{33} \times (\gamma^2 - \omega^2)]$$

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Received 28 January and accepted 14 January 2004